Chapter 9

Quadratics

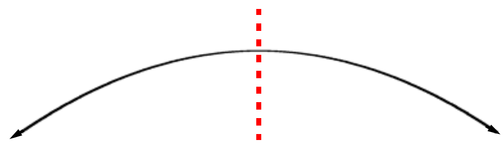
Section 9.1
Solving Quadratic Equations by Factoring

Similar to a *linear equation* ($y = mx + b$), a *quadratic equation* has two variables (x, y), but unlike in a linear equation, one of the variables is a square: $\boldsymbol{y = ax^2 + bx + c}$. *(quadratic = square)*

As the name indicates, a linear equation, when plotted, forms a line. On the other hand, a quadratic equation, when plotted, forms a curve.

A quadratic equation doesn't form just any curve, but a curve that is symmetrical (half of the curve is the reflection of the other half).

Called a parabola, it is the shape of the path of a baseball (or projectile) in flight.



The solution of a quadratic is achieved through factoring, if we remove the equal sign (=) and the y variable:

$$y = ax^2 + bx + c \longrightarrow ax^2 + bx + c$$

This is the type of trinomial studied in section 7.5. For example,

$x^2 - 2x - 8$ can be factored into $(x - 4)(x + 2)$

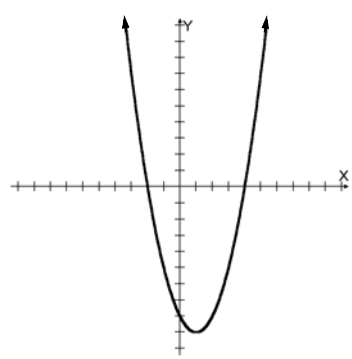
putting back the equal sign and the y variable it becomes: $y = (x - 4)(x + 2)$

This is the type of equation to be solved in this section.

HOW TO SOLVE A QUADRATIC EQUATION

Because these equations represent curves and curves come around (unlike the straight line of linear equations that crosses axes only once), in many cases a curve must cross the same axis twice. The figure to the right shows this.

A parabolic curve always comes back—like a boomerang. Notice how in the graph it comes down from left to right and crosses the x-axis twice before going off to the top right. Specifically, notice how the curve comes down and crosses the x-axis at –2, and comes back up crossing the x-axis again at +4. Factoring a quadratic equation, it is these very points we seek. These points become the "roots" or "zeros" of the quadratic.



The equation plotted in the previous graph is $y = x^2 - 2x - 8$

Because the value of y is zero ($y = 0$) when the parabola crosses the x-axis, we now set up the equation equal to zero to find the specific values of x when the curve crosses the x-axis (roots).

$$0 = x^2 - 2x - 8 \longrightarrow \text{Factoring the trinomial} \longrightarrow (x - 4)(x + 2) = 0$$

Because it is not known which of the two binomials, $(x - 4)$ or $(x + 2)$, is equal to zero, setting up each binomial equal to zero is the way to find both values for x:

$$x - 4 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = 4 \qquad\qquad x = -2$$

These answers are called the roots of the quadratic and can be verified in the graph above.

Example: Solve $(x + 2)(x - 3) = 0$

$(x + 2) = 0$ or $(x - 3) = 0$

$x = -2 \qquad x = 3$

Example: Solve $\left(\frac{1}{3}x - 2\right)\left(\frac{2}{5}x + 3\right) = 0$

$\frac{1}{3}x - 2 = 0$ or $\frac{2}{5}x + 3 = 0$

$\frac{1}{3}x = 2 \qquad \frac{2}{5}x = -3$

$x = 6 \qquad x = \frac{-15}{2} = -7.5$

Example: Solve $0 = x^2 + 7x + 12$

$0 = x^2 + 7x + 12$ (factor trinomial)

$0 = (x + 3)(x + 4)$

$x + 3 = 0 \qquad x + 4 = 0$

$x = -3 \qquad x = -4$

Example: Solve $x^2 - 5x - 4 = 2$

$x^2 - 5x - 4 = 2$ (move 2 to left-side and add to –4)

$x^2 - 5x - 6 = 0$ (factor)

$(x - 6)(x + 1) = 0$

$x - 6 = 0 \qquad x + 1 = 0$

$x = 6 \qquad x = -1$

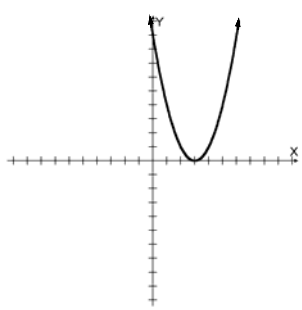
Example: Solve $x^2 - 6x = -9$

$x^2 - 6x + 9 = 0$ (perfect trinomial square)

$(x - 3)(x - 3) = 0$

$x - 3 = 0 \qquad x - 3 = 0$

$x = 3 \qquad x = 3$



One double, equal root, 3. This curve comes down and "touches" the x-axis only at 3, instead of crossing it.

Example: Solve $x^2 = 6x$

$x^2 - 6x = 0$ (find common factor)

$x(x - 6) = 0$

$x = 0 \qquad x - 6 = 0$

$x = 6$

Example: Solve $2x^2 + 9x + 10 = 0$

$2x^2 + 9x + 10 = 0$

$(2x + 5)(x + 2) = 0$

$2x + 5 = 0 \qquad x + 2 = 0$

$2x = -5 \qquad x = -2$

$x = -\frac{5}{2} = -2.5$

Example: Solve $6x^2 + x = 12$

$$6x^2 + x - 12 = 0$$
$$(2x + 3)(3x - 4) = 0$$
$$2x + 3 = 0 \qquad 3x - 4 = 0$$
$$2x = -3 \qquad 3x = 4$$
$$x = -\frac{3}{2} = 1.5 \qquad x = \frac{4}{3} = 1.\overline{3}$$

Practice:

CONTENTS OF THIS PAGE HAVE BEEN
REMOVED TO PROTECT COPYRIGHT

Section 9.2
Completing the Square

So far we have solved quadratic equations that can be factored, like:

$$x^2 - 5x - 6 = 0$$

Determine: Which two integers give a product of –6 and a difference of –5?

$$(x - 6)(x + 1) = 0$$

$$x - 6 = 0 \qquad\qquad x + 1 = 0$$

$$x = 6 \text{ and } -1$$

But what happens when we encounter a quadratic equation which cannot be factored?

Example: $x^2 - 4x - 7 = 0$

Determine: Which two integers give a product of –7 and a difference of –4?

The factors of 7 are 1 and 7

$$x^2 - 4x - 7 = 0$$

The differences of 7 and 1 are 6 or –6, not –4

Because 7 has only two factors (7 and 1), and these two factors subtracted are 6 or –6 and not –4, like the middle term above, the answers to this quadratic are not integers, but decimals. Finding which combination is correct would be a very lengthy and difficult task.

To solve this quadratic we turn to a process called, **completing the square**, which involves making a *perfect trinomial square* out of the left side of the equation.

$$x^2 - 4x - 7 = 0$$

STEP ONE: Move –7 to the right side of the equation, leaving an empty space where the –7 was:

$$x^2 - 4x \quad\quad = 7$$

STEP TWO: Take one-half of the middle term (–4) and square it. *Complete the trinomial square* with this number by placing it into the empty space (one-half of –4 is –2, –2 squared is 4).

Because 4 was added to the left-hand side, then 4 must be added to the right-hand side:

$$x^2 - 4x + 4 = 7 + 4$$

STEP THREE: Factor the left-hand side of the equation as a *perfect trinomial square*:

$$(x-2)(x-2) = 11$$
$$(x-2)^2 = 11$$

STEP FOUR: Find the square root of both sides: $\sqrt{(x-2)^2} = \sqrt{11} \longrightarrow x - 2 = \pm 3.317$

Answer: $x = 2 + 3.317 = \mathbf{5.317}$, $x = 2 - 3.317 = \mathbf{-1.317}$

HOW TO SOLVE A QUADRATIC WITH A LEADING COEFFICIENT GREATER THAN ONE

Example: Solve $2x^2 - 8x - 9 = 0$

Because to *complete a square* the leading coefficient MUST be one, divide the whole equation by 2:

$$\frac{2x^2}{2} - \frac{8x}{2} - \frac{9}{2} = 0 \longrightarrow x^2 - 4x - 4.5 = 0$$

Move –4.5 to the right: $x^2 - 4x = 4.5$

Complete the square: $x^2 - 4x + 4 = 4.5 + 4$

$x^2 - 4x + 4 = 8.5$

Factor: $(x-2)(x-2) = 8.5$

Find square root: $\sqrt{(x-2)^2} = \sqrt{8.5}$

$x - 2 = \pm 2.915$

$x = 2 + 2.915 = \mathbf{4.915}$ $x = 2 - 2.915 = \mathbf{-0.915}$

Example: Solve $3x^2 + 5x - 11 = 0$

Divide equation by 3 $\frac{3x^2}{3} + \frac{5x}{3} - \frac{11}{3} = 0$

Move $-\frac{11}{3}$ to the right $x^2 + \frac{5}{3}x = \frac{11}{3}$

Complete the square (one-half of $\frac{5}{3}$ is $\frac{5}{6}$ and $\frac{5}{6}$ squared is $\frac{25}{36}$) and add to both sides of the equation:

$$x^2 + \frac{5}{3}x + \frac{25}{36} = \frac{11}{3} + \frac{25}{36}$$

Add right-hand side $$x^2 + \frac{5}{3}x + \frac{25}{36} = \frac{157}{36}$$

Factor left-hand side as a *perfect trinomial square* and find square root of both sides:

$$\sqrt{\left(x + \frac{5}{6}\right)\left(x + \frac{5}{6}\right)} = \sqrt{\frac{157}{36}}$$

$$x + \frac{5}{6} = \pm 2.088$$

Solve for x: $x = -0.8\overline{3} + 2.088 = \mathbf{1.255}$ $\quad x = -0.8\overline{3} - 2.088 = \mathbf{-2.921}$

Practice:

CONTENTS OF THIS PAGE HAVE BEEN REMOVED TO PROTECT COPYRIGHT

Section 9.3
The Quadratic Formula

The quadratic formula is a way of solving all quadratic equations to find their *roots*. The quadratic formula is derived by the general use of *completing the square*.

If we take the general form of the quadratic equation $\boldsymbol{ax^2 + bx + c = 0}$

and use *completing the square* to solve for x, then the quadratic formula is born.

STEP ONE: Divide whole equation by a to reduce the leading coefficient to 1.

$$\frac{a}{a}x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \longrightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

STEP TWO: Move $\frac{c}{a}$ to the right side of the equation, leaving an empty space where the $\frac{c}{a}$ was:

$$x^2 + \frac{b}{a}x \qquad = -\frac{c}{a}$$

STEP THREE: Take one-half of the middle term $\left(\frac{b}{a}\right)$ and square it. *Complete the trinomial square* with this expression by placing it into the empty space (one-half of $\frac{b}{a}$ is $\frac{b}{2a}$, and $\frac{b}{2a}$ squared is $\frac{b^2}{4a^2}$).

Because $\frac{b^2}{4a^2}$ was added to the left-hand side, then $\frac{b^2}{4a^2}$ must be added to the right hand side:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

STEP FOUR: Factor the left hand-side of the equation as a *perfect trinomial square* and add the right-hand side:

FACTOR LEFT SIDE

$$\sqrt{x^2} = x \qquad \sqrt{\frac{b^2}{4a^2}} = \frac{b}{2a}$$

ADD RIGHT SIDE

$$-\frac{c}{a} + \frac{b^2}{4a^2} = \frac{-4ca + b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2} \quad \longrightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

STEP FIVE: Find the square root of both sides

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

STEP SIX: solve for x by moving $\frac{b}{2a}$ to the right hand side: $$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

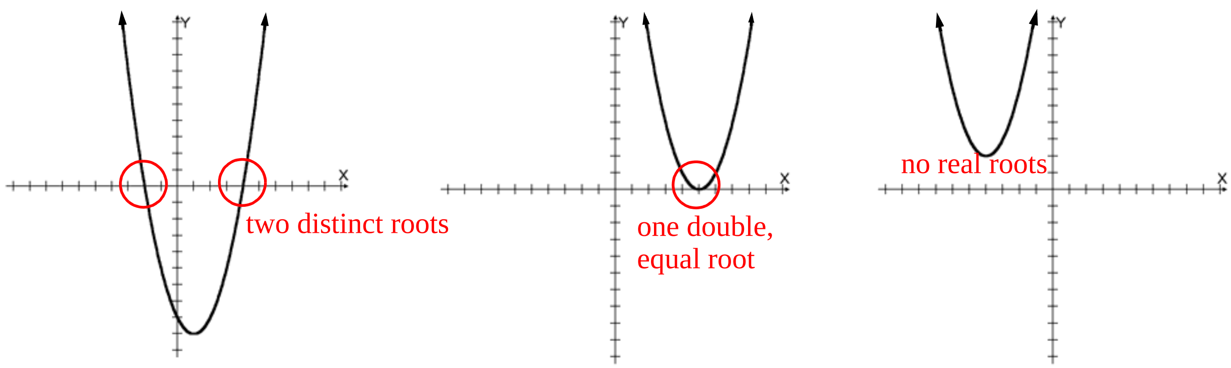
This is the formula for quadratic roots:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where ***a*** is the leading coefficient (quadratic term), ***b*** is the coefficient of the middle (linear) term, and ***c*** is the last (constant) number.

UNDERSTANDING WHAT THE ROOTS MEAN

In section 8.1 the shape of the quadratic line was given as the parabolic curve crossing the *x*-axis. Below left, the curve crosses the *x*-axis twice (two distinct roots), below center the curve does not cross but touches the *x*-axis (one double, equal root), below right the curve does not cross the *x*-axis (no real roots). Depending on the ***a***, ***b*** and ***c*** values of the quadratic formula, the shape and position of the curve is determined.



Example (for two distinct roots): Solve $2x^2 + 5x - 1 = 0$ $\quad$ $a = 2$ $\quad$ $b = 5$ $\quad$ $c = -1$

Substitute in the quadratic formula: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4(2)(-1)}}{2(2)} \rightarrow x = \frac{-5 \pm \sqrt{25 + 8}}{2(2)} \rightarrow x = \frac{-5 \pm \sqrt{33}}{4}$$

$$x = \frac{-5 + 5.745}{4} \rightarrow x = \frac{-5 - 5.745}{4}$$

$x = \mathbf{0.186}$ $\qquad$ $x = \mathbf{-2.686}$

Example (for one double, equal root): Solve $x^2 - 5x + 6.25 = 0$ $\quad$ $a = 1$ $\quad$ $b = -5$ $\quad$ $c = 6.25$

Substitute in the quadratic formula: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-5)\pm\sqrt{(-5)^2-4(1)(6.25)}}{2(1)} = x = \frac{5\pm\sqrt{25-25}}{2(1)} = x = \frac{5\pm\sqrt{0}}{2} = x = \frac{5}{2} = x = \mathbf{2.5}$$

Example (for no roots): Solve $x^2 + 2x + 6 = 0$ $\quad$ $\boldsymbol{a} = 1$ $\quad$ $\boldsymbol{b} = 2$ $\quad$ $\boldsymbol{c} = 6$

Substitute in the quadratic formula: $\quad x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x = \frac{-2\pm\sqrt{(2)^2-4(1)(6)}}{2(1)} \rightarrow x = \frac{-2\pm\sqrt{4-24}}{2} \rightarrow x = \frac{-2\pm\sqrt{-20}}{2}$$

Because the square root of a negative number cannot be determined, the answer is not real.

The Discriminant: $\sqrt{b^2-4ac}$

The discriminant is that part of the quadratic formula that allows us to peek into it to see if the quadratic has any valid roots. If the discriminant is negative, no need to continue; if it is positive or zero, continue.

Example: Solve $7x^2 - 13x + 13 = 0$ $\quad$ $\boldsymbol{a} = 7$ $\quad$ $\boldsymbol{b} = -13$ $\quad$ $\boldsymbol{c} = 13$

Using the discriminant $\sqrt{(-13)^2-4(7)(13)} = \sqrt{169-364} = \sqrt{-195}$ Radical is negative, zero roots.

Practice:

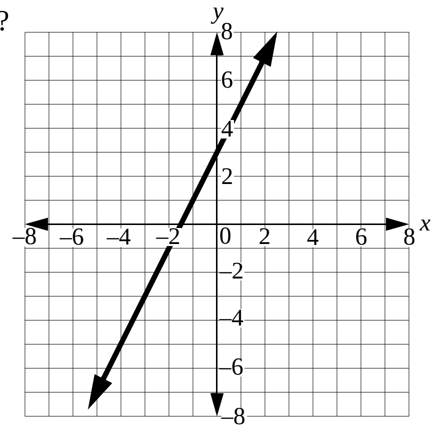
Section 9.4
Exponential Functions (Growth and Decay)

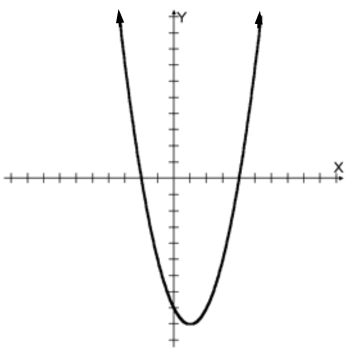
What shape would a line on a graph have if the exponent is a variable?

From past experience we know that an equation such as

$$y = 2x + 3$$

is a straight line (see graph at right).





Also from past experiences we discovered that squaring the base "x", will turn the line into a curve,

$$y = x^2 – 2x –8$$

specifically a parabola (see graph to the left).

What shape would a line on a graph have if the exponent is a variable?

The shape is a curve that increases (climbs) rapidly. Naturally, we call this type of curve "exponential growth." The particular equation of the graph in figure 1 is:

$$y = 2^x$$

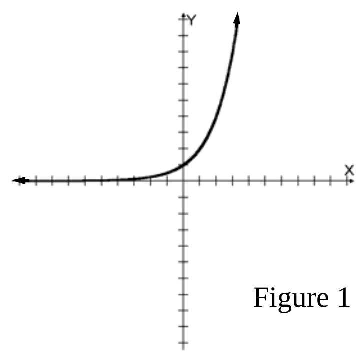


Figure 1

VARIATIONS OF THE EXPONENTIAL CURVE

Because the coefficient is negative, a curve such as $y = –2^x$ (figure 2)

will send the curve the opposite way, towards the negative side.

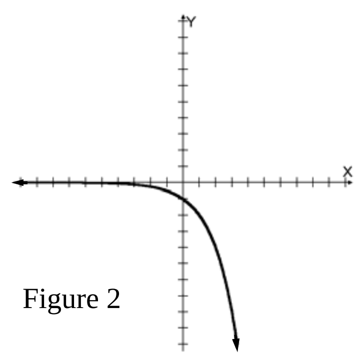


Figure 2

A decimal base such as $y = 0.5^x$

will make the curve decrease (drop) rapidly. This type of curve is referred to as "exponential decay." See figure 3.

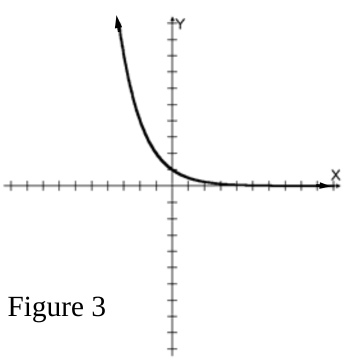


Figure 3

Example in exponential growth:

On her birthday, a five-year-old receives a $5,000 gift and her parents place the money in a bank account that earns interest of 7% per year (7 cents for very dollar deposited). How much money would be available in the account when the girl turns 20 (15 years later)?

After the first year one dollar would have grown to $1.07, and after the second year the $1.07 would have to grow to $1.1449 ($1.07 \times 1.07$), and after the third year to $1.225, and so on.

This problem fits the exponential growth pattern because the more interest is accumulated, the higher the interest earned. In other words, the money grows rapidly by compounding the interest amount. The exponential equation would be:

$$y = a(1 + i)^x$$

Where y is the final amount
a is the initial amount
i is the rate (percent) of increase per period in decimal form
x is the number of periods (years)

Solving for y $\quad y = 5000\,(1 + 0.07)^{15}$

Because multiplying 1.07 to the 15th power is tedious, we use the y^x function of a scientific calculator*:

Press 1.07
Press y^x
Press 15
Press =

$$(1 + 0.07)^{15} = 2.759031541$$

$$y = 5000\,(2.759031541) = 13795.16$$

The girl at twenty will have $13,795.16. She earned $8,795.16 in interest.

Example in exponential decay:

A chlorine solution decays 50% (0.5) for every week it is left uncovered. If the original amount of chlorine used is 2560 ounces (20 gallons), how much chlorine is available in the solution after 12 weeks?

If it decays 50% (half the amount) every week, after the first week there is $\frac{2560}{2} = 1280$ ounces, after the second week there is $\frac{1280}{2} = 640$ ounces available, and so on.

This problem fits the exponential decay pattern because the amount of chlorine drops rapidly. The exponential equation would be

$$y = a(1 - i)^x$$

Compare this equation to the one above. For growth, i is added, for decay, i is subtracted.

Where y is the final amount
a is the initial amount
i is the rate (percent) of decrease per period in decimal form
x is the number of periods (weeks)

Solving for y $y = 2560\,(1 - 0.5)^{12}$

Because multiplying 0.5 to the 12th power is tedious, we use the y^x function of a scientific calculator*:

Press 0.5
Press y^x
Press 12
Press =

$$(1 - 0.5)^{12} = (0.5)^{12} = 0.000244141$$

$$y = 2560\,(0.000244141) = 0.625$$

The amount of available chlorine after 12 weeks is 0.625 ounces (slightly more than a tablespoon).

*Performed on a TI-35X calculator.

Practice:

CONTENTS OF THIS PAGE HAVE BEEN REMOVED TO PROTECT COPYRIGHT