Chapter 7

Factoring

Section 7.1
Common Factors in Polynomials

In mathematics, factors are the parts that make a whole, and factoring involves division because they are the numbers you get when you divide a number or expression (a given quantity) without a remainder.

The FACTORS of numbers are always smaller than the numbers.

Example: If we divide $\frac{6}{2} = 3$, the factors of **6** are **2** and **3**.

To find ALL the factors of a number (called *prime factorization*), divide the number by each prime number starting with the number 2, until you can't divide again without getting a remainder (decimal.)

Example: Find all the factors of 54

$$\frac{54}{2} = \frac{27}{3} = \frac{9}{3} = 3$$

Answer: The factors of 54 are (2)(3)(3)(3) = $(2)(3^3)$

Example: Factor 18 + 36 + 54 completely

The prime factorization of 18 is (2)(3)(3)
The prime factorization of 36 is (2)(2)(3)(3)
The prime factorization of 54 is (2)(3)(3)(3)

2 is common to all three numbers once: 2
3 is common to all three numbers twice: (3)(3) = 9

(9)(2) = 18

Because the highest common factor for all three numbers is 18, the expression 18 + 36 + 54 can be factored to:

18(1 + 2 + 3)

By dividing $\frac{18}{18}$, $\frac{36}{18}$, and $\frac{54}{18}$, 18 has been factored to show that 18 + 36 + 54 can also be written as the product of

18(1 + 2 + 3)

In algebra, factoring works the same way, except this time with numbers AND symbols.

Example: Factor $18a^3 + 36a^2 + 54a$ completely

The factors of of $18a^3$ are $(2)(3)(3)(a)(a)(a)$
The factors of of $36a^2$ are $(2)(2)(3)(3)(a)(a)$
The factors of of $54a$ are $(2)(3)(3)(3)(a)$

Common in all three terms we find $(2)(3)(3)(a) = 18a$

Because the highest common factor for all three numbers is $18a$, the expression $18a^3 + 36a^2 + 54a$ can be factored: $18a\,(a^2 + 2a + 3)$

By dividing $\frac{18a^3}{18a} = a^2$, $\frac{36a^2}{18a} = 2a$, and $\frac{54a}{18a} = 3$, $18a$ has been factored to show that:

$18a^3 + 36a^2 + 54a$ can also be written as the product of $18a\,(a^2 + 2a + 3)$

Example: Factor completely $4x^6 + 6x^4 - 14x^3 + 2x^2$

The common coefficient found in 4, 6, 14, and 2 is **2**.
The common base found ***x***.
The common exponent found is **2**.

The common factor is then $\mathbf{2x^2}$.

By dividing each term of the expression by $\mathbf{2x^2}$

$4x^6$ becomes $2x^4$
$6x^4$ becomes $3x^2$
$14x^3$ becomes $7x$
$2x^2$ becomes 1

Answer: $2x^2\,(2x^4 + 3x^2 - 7x + 1)$

Practice:

CONTENTS OF THIS PAGE HAVE BEEN REMOVED TO PROTECT COPYRIGHT

Section 7.2
Difference of Two Squares

A *difference of two squares* is a binomial that can be factored into two other binomials. Because a square, by definition, is the product of two identical numbers (for instance, 16 is the square of 4 × 4, 9 is the square of 3 × 3), the *difference of two squares* is two squares with a subtraction sign between them.

Examples: $x^2 - 4$ is a *difference of squares* because both x^2 and 4 are squares.

$9y^2 - 25$ is a *difference of squares* because 9, y^2 and 25 are squares.

FACTORING A DIFFERENCE OF TWO SQUARES

To factor a *difference of two squares*, set up two binomials in parenthesis: One separated by addition, another one separated by subtraction. Place the square roots of the first square to start each parenthesis, and the square roots of the second square to end each parenthesis. The first example above, $x^2 - 4$, is factored into

$(x + 2)(x - 2)$ because the $\sqrt{x^2} = x$ and the $\sqrt{4} = 2$

The factors of the second example above, $9y^2 - 25$, are $(3y + 5)(3y - 5)$

because the $\sqrt{9} = 3$, the $\sqrt{y^2} = y$, and the $\sqrt{25} = 5$

Example: Factor $49a^2 - 100$

Answer: $(7a + 10)(7a - 10)$ because the $\sqrt{49} = 7$, the $\sqrt{a^2} = a$, and the $\sqrt{100} = 10$

Check work by multiplying the two factored binomials. Multiplication takes us back to the original binomial:

Multiply $(7a + 10)(7a - 10)$

using F.O.I.L. $(7a)(7a) + (7a)(-10) + (10)(7a) + (10)(-10)$

$49a^2 - 70a + 70a - 100$

cancelling $-70a + 70a$, we get back the original $49a^2 - 100$

FACTORING MORE THAN ONCE

Factoring **completely** means that in some occasions factoring is not finished on the first try and some more factoring should be done.

Example: Factor $x^4 - 16$

Because it is a difference of squares, you factor it into two binomials (one addition and one subtraction):

$$(x^2 + 4)(x^2 - 4)$$

The first binomial is a "sum of squares" (black) and cannot be factored. The second binomial (red) is another *difference of two squares*, thus turning the "complete factoring" into:

$$(x^2 + 4)(x^2 - 4) \longrightarrow (x^2 + 4)(x + 2)(x - 2)$$

Example: Factor $x^8 - 1$

First round	$(x^4 + 1)(x^4 - 1)$
Second round	$(x^4 + 1)(x^2 + 1)(x^2 - 1)$
Third round	$(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)$

Practice:

CONTENTS OF THIS PAGE HAVE BEEN REMOVED TO PROTECT COPYRIGHT

Section 7.3
Perfect Trinomial Squares

Like the *difference of two squares*, a *perfect trinomial square* is formed by two squares, but unlike the *difference of two squares*, it's a trinomial and not a binomial.

HOW TO RECOGNIZE A TRINOMIAL THAT IS A PERFECT SQUARE

—The last term must be positive
—The first and third term (corners) must be squares
—The middle term is twice the product of the square roots of the first and last terms (corners).

Examples: $x^2 + 2x + 1$ is a *perfect trinomial square* because the last term is positive

the first and last terms are squares ($\sqrt{x^2} = x$ and $\sqrt{1} = 1$)

the middle term is $2(x \times 1) = 2x$

twice

$4x^2 + 12x + 9$ is a *perfect trinomial square* because the last term is positive

the first and last terms are squares ($\sqrt{4x^2} = 2x$ and $\sqrt{9} = 3$)

the middle term is $2(2x \times 3) = 12x$

twice

$25y^2 - 45y - 36$ is NOT a *perfect trinomial square* because the last term is negative.

$16a^2 + 30a + 9$ is NOT a *perfect trinomial square* because the middle term is not twice the product of the square roots of the first and last terms $2(4a \times 3) = 24a$

FACTORING A PERFECT TRINOMIAL SQUARE

To factor a *perfect trinomial square*, set up two binomials in parentheses. If the middle term is addition, both binomials are additions, if the middle term is subtraction, both binomial are subtractions. Find the square root of the first term and third term. Place the square roots of the first square to start each parenthesis, and the square roots of the second square to end each parenthesis. Because the middle term is addition, the first example above, $x^2 + 2x + 1$ is factored into

$$(x + 1)(x + 1)$$

Because both binomials are identical, square them to: $(x + 1)^2$

The second example above $4x^2 + 12x + 9$ can be factored to $(2x + 3)(2x + 3) = (2x + 3)^2$

Example: Factor $25x^2 - 60xy + 36y^2$

First term: $\sqrt{25x^2} = 5x$ second term: $\sqrt{36y^2} = 6y$ middle term: $(2)(5x \times 6y) = 60xy$

It is a perfect trinomial square. Because the middle term of the original trinomial is subtraction, both binomials will be subtractions.

Answer: $(5x - 6y)(5x - 6y) = (5x - 6y)^2$

Practice:

CONTENTS OF THIS PAGE HAVE BEEN
REMOVED TO PROTECT COPYRIGHT

Section 7.4
Factoring Trinomials: ($x^2 + bx + c$)

A trinomial with a leading coefficient of one is factored by creating two binomials in parentheses and determining the correct addition or subtraction (to match the middle term) of the products of the two inside and two outside values that form the two binomials.

Sum of 7 → Product of 12

Example: Factor the trinomial $x^2 + 7x + 12$

Determine: Which two integers (when multiplied) give a product of 12 and (when added) a sum of 7?

There are only three possible multiplications that yield a 12: 4×3 6×2 12×1

Of these three, the correct sum is the one that adds up (3 + 4) to 7. Answer: $(x + 3)(x + 4)$

Using F.O.I.L. to check the answer:

—First: $(x)(x) = x^2$
—Outside: $(x)(4) = 4x$
—Inside: $(3)(x) = 3x$
—Last: $(3)(4) = 12$

Add like terms
$3x + 4x = 7x$

$x^2 + 7x + 12$ (check!)

Note: When the last term is positive and the middle term is negative, then both binomials are subtractions.

Example:
Factor the above example with a negative middle term.

$x^2 - 7x + 12$ Answer: $(x - 3)(x - 4)$

Using F.O.I.L. to check the answer:

—First: $(x)(x) = x^2$
—Outside: $(x)(-4) = -4x$
—Inside: $(-3)(x) = -3x$
—Last: $(-3)(-4) = 12$

Add like terms
$-3x + -4x = -7x$

$x^2 - 7x + 12$ (check!)

When to add, when to subtract
Whether we add or subtract to form the two binomials is **determined by the sign of the last term**. If the sign of the last term is positive, the factors of the product of the last term are added. If the sign of the last term is negative, the factors of the product of the last term are subtracted.

Example: Factor $y^2 + y - 20$

Determine: Which two integers (if multiplied) give a product of –20 and (if subtracted) a difference of 1?

Possible factors: -1×20 -2×10 -4×5

only the difference of $5 - 4 = 1$ fits Answer: $(y - 4)(y + 5)$

Example: Factor $y^2 - 5y - 6$

Determine: Which two integers give a product of –6 and a difference of –5?

Possible factors: -6×1 -2×3

only the difference of $1 - 6 = -5$ fits Answer: $(y - 6)(y + 1)$

Example: Factor $y^2 - 11y + 18$

Determine: Which two integers give a product of 18 and a sum of –11?

Possible factors: $(-18)(-1)$ $(-2)(-9)$ $(-3)(-6)$

only the sum of $-2 - 9 = -11$ fits Answer: $(y - 9)(y - 2)$

Practice:

CONTENTS OF THIS PAGE HAVE BEEN
REMOVED TO PROTECT COPYRIGHT

Section 7.5
Factoring Trinomials: ($ax^2 + bx + c$)

The way to factor a trinomial with a leading coefficient greater than one is similar to factoring a trinomial with a leading coefficient of one, except that now you also have to consider the factors of the product of the leading (first term) coefficient.

Example: Factor $2x^2 + 9x + 10$

Because the last term is positive, the two binomials will be additions.

Determine: Which two integers give a product of 10 and a sum of 9, considering that the first factors must give a product of 2?

Possible first factors: 2×1

Possible last factors: 10×1 5×2

Possible combinations:
$(2x + 10)(x + 1)$
$(2x + 1)(x + 10)$
$(2x + 5)(x + 2)$ Correct!
$(2x + 2)(x + 5)$

When using F.O.I.L., all combinations yield the first and last term of the trinomial correctly, but only the third one the correct middle term.

$2x^2 + 4x + 5x + 10$ $2x^2 + 9x + 10$

Answer: $(2x + 5)(x + 2)$

Example: Factor $4x^2 – 4x – 15$

Because the last term is negative, one binomial will be addition, the other one will be subtraction.

Determine: Which two integers give a product of –15 and a difference of –4, considering that the first factors must give a product of 4?

Possible first factors: 2×2 4×1

Possible last factors: $–15 \times 1$ $–5 \times 3$

Possible combinations:

1. $(2x – 15)(2x + 1)$
2. $(2x – 5)(2x + 3)$ ← Correct!
3. $(4x – 15)(x + 1)$
4. $(4x + 1)(x – 15)$
5. $(4x + 5)(x – 3)$
6. $(4x + 3)(x – 5)$

When using F.O.I.L., all combinations yield the first and last term of the trinomial correctly, but only the second one has the correct middle term.

Answer: $(2x – 5)(2x + 3)$

Example: Factor $6x^2 + x - 12$

Because the last term is negative, one binomial will be addition, the other one will be subtraction.

Determine: Which two integers give a product of –12 and a difference of 1, considering that the first factors must give a product of 6?

Possible first factors: 6×1 and 2×3

Last possible factors: 12×1 6×2 4×3

Possible combinations:

1. $(6x + 12)(x - 1)$	**7**. $(2x + 12)(3x - 1)$	
2. $(6x - 1)(x + 12)$	**8**. $(2x - 1)(3x + 12)$	
3. $(6x - 6)(x + 2)$	**9**. $(2x + 6)(3x - 2)$	
4. $(6x - 2)(x + 6)$	**10**. $(2x - 2)(3x + 6)$	
5. $(6x - 4)(x + 3)$	**11**. $(2x + 4)(3x - 3)$	
6. $(6x - 3)(x + 4)$	**12**. $(2x + 3)(3x - 4)$	Correct!

When using F.O.I.L., all combinations yield the first and last term of the trinomial correctly, but only the twelfth one produces the correct middle term ($-8x + 9x = x$).

Answer: $(2x + 3)(3x - 4)$

Practice:

CONTENTS OF THIS PAGE HAVE BEEN REMOVED TO PROTECT COPYRIGHT

Section 7.6
Factoring by Grouping

Factoring by grouping is a way of breaking down an expression into more factorable expressions.

Example: Factor $x^3 + x^2 + x + 1$

The expression above does not have a common factor; it is not a *difference of squares* and it is not a *trinomial*. However, the expression can be separated into two binomials.

$$(x^3 + x^2) + (x + 1)$$

Factoring x^2 from the left binomial: $x^2(x + 1)$

The new expression becomes: $x^2(x + 1) + (x + 1)$ or $x^2(x + 1) + 1(x + 1)$

Notice now that the binomial $(x + 1)$ is a common factor within the brackets.

Factoring again, this time $(x + 1)$: $(x + 1)(x^2 + 1)$

Example: Factor $12y^3 - 4y^2 + 15y - 5$

Splitting in two binomials $(12y^3 - 4y^2) + (15y - 5)$

Factoring binomials separately $4y^2(3y - 1) + 5(3y - 1)$

Factoring the common factor $(3y - 1)$: $(3y - 1)(4y^2 + 5)$

Example: Factor $a^2c^2 + bc^2 - 2a^2 - 2b$

Splitting in two binomials $(a^2c^2 + bc^2) + (-2a^2 - 2b)$

Factoring binomials separately (notice that in the second binomial the negative sign was factored also).

$$c^2(a^2 + b) - 2(a^2 + b)$$

Factoring the common factor $(a^2 + b)$: $(a^2 + b)(c^2 - 2)$

Example: Factor $4x^2 - 12xy + 9y^2 - z^2$

In this particular example, splitting the polynomial in two binomials will serve no purpose; however, because there is a *perfect trinomial square* and a negative square at the end, the split will leave a trinomial and $(-z^2)$.

$$(4x^2 - 12xy + 9y^2) - z^2$$

Factoring the trinomial: $(2x - 3y)(2x - 3y) = (2x - 3y)^2$

The *perfect trinomial square* becomes the first term of a *difference of two squares*, where $\mathbf{z^2}$ is the second term

$$(2x - 3y)^2 - \mathbf{z^2}$$

Factoring the *difference of two squares:* $[(2x - 3y) + \mathbf{z}\,][(2x - 3y) - \mathbf{z}]$

Eliminating parentheses: $[2x - 3y + \mathbf{z}\,][2x - 3y - \mathbf{z}]$

Example: Factor $x^6 - x^4 - x^2 + 1$

Splitting in two binomials $(x^6 - x^4) + (-x^2 + 1)$

Factoring binomials separately (notice that in the second binomial the negative sign was factored also).

$$[x^4(x^2 - 1)] - [1(x^2 - 1)]$$

Factoring the common factor $(x^2 - 1)$: $(x^2 - 1)(x^4 - 1)$

Factor both binomials (*difference of two squares*): $(x + 1)(x - 1)(x^2 + 1)(x^2 - 1)$

Last term is still a *difference of two squares*. Continue to factor: $(x + 1)(x - 1)(x^2 + 1)(x + 1)(x - 1)$

Rearrange in descending order and square like binomials: $(x^2 + 1)(x + 1)^2(x - 1)^2$

Practice: