Chapter 6

Polynomials

Section 6.1
Monomials: Multiplication and Division

In its most simple form, a monomial is a single combination of *coefficient*, *base*, and *exponent*. Also called a term, an example of a monomial would be the expression

coefficient → $2x^3$ ← exponent (3), ← base (x)

where x is the base, 2 the coefficient of x, and 3 the exponent.

For practical purposes, when the coefficient or exponent is one (1), the ONES are there, but we do not show them. Thus, if you write

$$x \text{ it means } 1x^1$$

The coefficient of x is 1 and the exponent is 1

MULTIPLYING MONOMIALS

Every time we multiply the coefficients of monomials, the exponents of identical bases must be added.

Example:

exponents: 4 + 5 = 9

$$(3x^4)(4x^5) = 12x^9$$

coefficients: (3)(4) = 12

If the bases are not the same, we may multiply the coefficients, but we cannot add the exponents.

Example:

exponents: 4 and 5 cannot be added

$$(3x^4)(4y^5) = 12x^4y^5$$

coefficients: (3)(4) = 12

Example: Multiply $(5x^2)(2xy)(3y^5) = 30x^3y^6$

Multiplying the coefficients (5 × 2 × 3) produces an answer of 30. Adding the exponents (2 + 1 = 3), the answer is 3 for x, and (1 + 5 = 6) for y.

DIVIDING MONOMIALS

Because multiplication and division are inverse operations, in division exponents get subtracted.

Example:

6 – 4 = 2

24 ÷ 3 = 8

$$\frac{24x^6}{3x^4} = 8x^2$$

Example: $$\frac{18x^2y^5z^7}{9xy^2} = 2xy^3z^7$$

Dividing the coefficients 18 and 9, we get 2. Subtracting the exponents, we get (2 –1 = 1) for *x*, and (5 – 2 = 3) for *y*. The exponent for *z* stays the same

Another way of looking at division

Example: $$\frac{21a^8b^6c^2}{3a^{10}b^2c^4} = \frac{21aaaaaaaabbbbbbcc}{3aaaaaaaaaabbcccc} = \frac{7b^4}{a^2c^2}$$

21 divided by 3 is 7. The exponents of *a* are 8 in the numerator and 10 in the denominator. Cancelling them leaves 2 in the denominator only. For *b*, the exponents are 6 in the numerator and 2 in the denominator; cancelling them leaves 4 in the numerator. For *c*, like for *a*, there are more of them in the denominator than in the numerator, therefore, the balance of c^2 is found in the denominator.

Practice:

CONTENTS OF THIS PAGE HAVE BEEN REMOVED TO PROTECT COPYRIGHT

Section 6.2
Polynomials

Polynomials are algebraic expressions with two or more terms separated by either an addition or a subtraction sign.

—A two-term polynomial is called a **binomial**
—A three-term polynomial is called a **trinomial**

Example: $3x^5 + 5x^2$ is a binomial $4y^3 + 2y^5 - y^6$ is a trinomial

We can write polynomials in descending order according to the value of the exponent

$$3x^7 + 2x^6 - 5x^5 + x^4 - 4x^3$$

Or in ascending order according to the value of the exponent

$$-4x^3 + x^4 - 5x^5 + 2x^6 + 3x^7$$

DEGREE OF THE POLYNOMIAL

The degree of the polynomial is established by the largest exponent found in any one term of the polynomial. If the term has more than one variable, we add the exponents of that term to determine the degree of the polynomial.

Example: The degree of the polynomial $x^6 + 5x^4 + 3x^2 + 7x$ is 6 because the largest exponent is 6.

Example: The degree of the polynomial $\overset{7}{x^4y^3} + \overset{6}{3x^3y^3} + \overset{5}{2x^3y^2} + \overset{3}{4xy^2} + \overset{2}{xy}$ is 7 because the highest sum [4 + 3 = 7] of the exponents of the polynomial is 7.

To raise an exponential expression to a power, multiply exponents

Example: $(x^a)^b = x^{a \times b}$ or $(x^2)^3 = x^{2 \times 3} = x^6$ or $(2^2)^3 = 2^{2 \times 3} = 2^6 = 64$

Example: $(3x^4)^3$ All the contents of the parenthesis are raised. $3 \cdot 3 \cdot 3x^{4 \times 3} = 27x^{12}$

NEGATIVE EXPONENTS AND ZERO EXPONENTS

Negative exponents, like positive exponents, may be written in descending order.

$$x^5 + x^4 + x^3 + x^2 + x + x^0 + x^{-1} + x^{-2} + x^{-3} + x^{-4} + x^{-5}$$

Because the division of two identical polynomials cancel each other, all polynomials to the zero (0) power are equal to 1. $2 - 2 = 0 \rightarrow \dfrac{x^2}{x^2} = x^0 = 1$

Negative exponents are positive in the reciprocal. $x^{-4} = \frac{1}{x^4}$

Therefore, a descending list of monomials with negative exponents

$$x^5 + x^4 + x^3 + x^2 + x + x^0 + x^{-1} + x^{-2} + x^{-3} + x^{-4} + x^{-5}$$

may be written $x^5 + x^4 + x^3 + x^2 + x + x^0 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5}$

Using negative exponents
Using the above expression and substituting 2 for x, the resulting polynomial would be:

$$2^5 + 2^4 + 2^3 + 2^2 + 2 + 2^0 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} \quad \textbf{OR} \quad 32 + 16 + 8 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$$

Example: Multiply the monomials $(y^4)(2y^2)(4y^{-3})$

$4 + 2 - 3 = 3$ (exponents)

$$(y^4)(2y^2)(4y^{-3}) = 8y^3$$

$1 \times 2 \times 4 = 8$ (coefficients)

Example: Divide the monomials $\frac{x^5y^3z^{-2}}{x^2y^6z}$

$$\frac{x^5y^3z^{-2}}{x^2y^6z} = x^3y^{-3}z^{-3}$$

For x: $5 - 2 = 3$

For y: $3 - 6 = -3$

For z: $-2 - 1 = -3$

Answer: Make negative exponents positive $\frac{x^3}{y^3z^3}$

The answer may also be reached by solving negative exponents another way. In this way we first fix the negative exponent (making it positive in the reciprocal) and then reduce them like in a fraction.

$$\frac{x^5y^3z^{-2}}{x^2y^6z} = \frac{x^5y^3}{x^2y^6zz^2} = \frac{xxxxxyyy}{xxyyyyyyzzz} = \frac{xxx}{yyyzzz} = \frac{x^3}{y^3z^3}$$

Practice:

CONTENTS OF THIS PAGE HAVE BEEN
REMOVED TO PROTECT COPYRIGHT

Section 6.3
Addition and Subtraction of Polynomials

In polynomials, algebraic addition and subtraction follow similar rules, and what is important in algebraic addition is the number line (see section 1.6). In the number line we compare negative and positive numbers and the answer is determined by the balance found.

Example: $8x + 9x - 3x - 7x + 2x = 9x$

To solve the above expression using the number line

- starting at zero
- move 8 to the right = 8
- move 9 to the right = 17
- move 3 to the left = 14
- move 7 to the left = 7
- move 2 to the right = 9

Example: $-12y + 5y + 2y - 6y + 16y - 7y = -2y$

Using the number line

- starting at zero
- move 12 to the left = –12
- move 5 to the right = –7
- move 2 to the right = –5
- move 6 to the left = –11
- move 16 to the right = 5
- move 7 to the left = –2

ADDITION OF POLYNOMIALS

Only the coefficients of monomials with the same base and the same exponent may be added or subtracted. The expression

$$3a^2 + a$$

cannot be added. However, the expressions

$3a^2 + a^2 = 4a^2$ OR $3a + a = 4a$ may be added because they have the same base and exponent

Example: Evaluate $3x + 5x - x + 6x = 13x$ Combine coefficients $3 + 5 - 1 + 6 = 13$

$$3x + 5x - x + 6x = 13x$$

Example: Evaluate: $(3x^3 + 2x^2 - 5x) + (5x^3 + x^2 - 2x)$

Remove parentheses $3x^3 + 2x^2 - 5x + 5x^3 + x^2 - 2x$

Combine coefficients according to base and exponent

$3x^3 + 5x^3 = 8x^3$

$2x^2 + x^2 = 3x^2$ → Answer: $8x^3 + 3x^2 - 7x$

$-5x - 2x = -7x$

Example: $(-a^7 + 6a^4 - 4a^3) + (-5a^4 + 3a^3 - 8a^2)$

remove parentheses $-a^7 + 6a^4 - 4a^3 - 5a^4 + 3a^3 - 8a^2$

Combine coefficients according to base and exponent

$-a^7$

$6a^4 - 5a^4 = a^4$ → Answer: $-a^7 + a^4 - a^3 - 8a^2$

$-4a^3 + 3a^3 = -a^3$

$-8a^2$

Practice:
Add algebraically.

SUBTRACTION OF POLYNOMIALS

To find the answer to a polynomial subtraction, turn the algebraic subtraction into algebraic addition and proceed like in addition. We do this by changing signs first.

Because polynomial subtractions must be encased in a parenthesis, this is done by changing **every** sign inside the polynomial.

Example: $(3x^3 + 2x^2 - 5x) - (5x^3 + x^2 - 2x)$

Remove parentheses. Notice how the whole trinomial in the second parenthesis changes signs

$$3x^3 + 2x^2 - 5x - 5x^3 - x^2 + 2x$$

Proceed with addition

$3x^3 - 5x^3 = -2x^3$

$2x^2 - x^2 = x^2$

$-5x + 2x = -3x$

Answer: $-2x^3 + x^2 - 3x$

Example: $(-a^7 + 6a^4 - 4a^3) - (-5a^4 + 3a^3 - 8a^2)$

remove parentheses $-a^7 + 6a^4 - 4a^3 + 5a^4 - 3a^3 + 8a^2$

Combine coefficients according to base and exponent

$-a^7$

$6a^4 + 5a^4 = 11a^4$

$-4a^3 - 3a^3 = -7a^3$

$8a^2$

Answer: $-a^7 + 11a^4 - 7a^3 + 8a^2$

Practice:

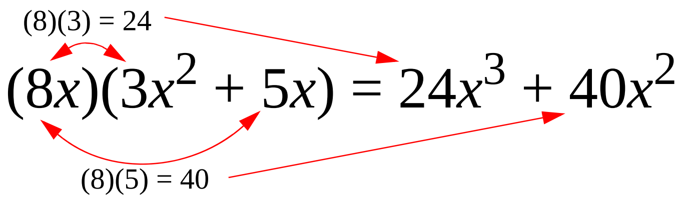
Subtract algebraically.

Section 6.4
Multiplication of Polynomials

MULTIPLYING A MONOMIAL BY A BINOMIAL

Similar to multiplication of integers, in the multiplication of polynomials we use the distributive property: Each term of the polynomial is multiplied by the monomial.

Example: Multiply $(8x)(3x^2 + 5x)$



Notice that the rules of multiplication of monomials have been applied: The exponents have been added (1 + 2 = 3) in the first product and (1 + 1 = 2) in the second one.

MULTIPLYING BINOMIALS

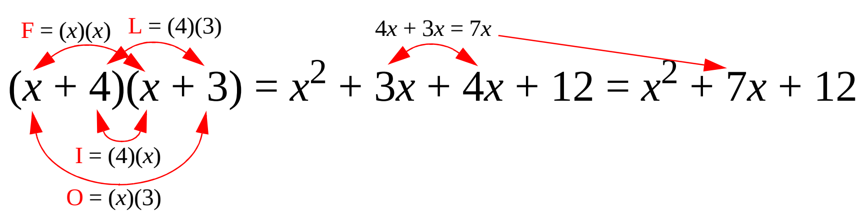
Multiplying binomials is often called the F.O.I.L. method, where F stands for *first* terms, O for *outside* terms, I for *inside* terms, and L for *last* terms.

Example: Multiply $(x + 4)(x + 3)$

See how the F.O.I.L. method is used:

- First: $(x)(x) = x^2$
- Outside: $(x)(3) = 3x$
- Inside: $(4)(x) = 4x$
- Last: $(4)(3) = 12$

Add like terms
$3x + 4x = 7x$



Example: Multiply $(3a + 5b)(2a - b)$

- First: $(3a)(2a) = 6a^2$
- Outside: $(3a)(-b) = -3ab$
- Inside: $(5b)(2a) = 10ab$
- Last: $(5b)(-b) = -5b^2$

Add like terms
$-3ab + 10ab = 7ab$

$$(3a + 5b)(2a - b) = 6a^2 - 3ab + 10ab - 5b^2 = 6a^2 + 7ab - 5b^2$$

SPECIAL PRODUCTS OF BINOMIALS

Special binomials are binomials that look similar. However, because the positive and negative signs are positioned differently, their products change considerably. They are:

1. $(a + b)(a + b)$
2. $(a - b)(a - b)$
3. $(a + b)(a - b)$

The product of the first one above (using FOIL) is:

$(a)(a) + (a)(b) + (a)(b) + (b)(b) = a^2 + ab + ab + b^2 = \mathbf{a^2 + 2ab + b^2}$

The product of the second one is:

$(a)(a) - (a)(b) - (a)(b) + (b)(b) = a^2 - ab - ab + b^2 = \mathbf{a^2 - 2ab + b^2}$

The product of the third one is:

$(a)(a) - (a)(b) + (a)(b) - (b)(b) = a^2 - ab + ab + b^2 = \mathbf{a^2 - b^2}$ ← *difference of two squares*

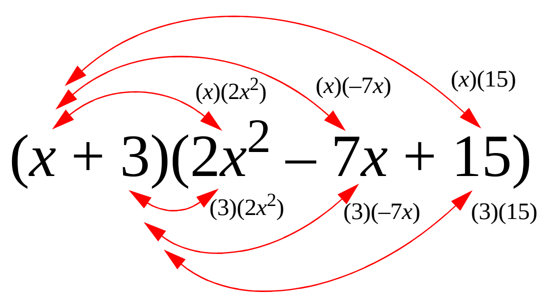
Notice that in the multiplication of the third one, $-ab$ and $+ab$ cancel out. Because of this, this particular "special product" is called *the difference of two squares*.

NOTE: The *sum of two squares*, $a^2 + b^2$, is not the product of any multiplication.

MULTIPLYING A BINOMIAL AND A TRINOMIAL

We continue to multiply each term in the first polynomial by every other term in the second polynomial.

Example: Multiply $(x + 3)(2x^2 - 7x + 15)$



First term of binomial times trinomial: $(x)(2x^2) + (x)(-7x) + (x)(15) = 2x^3 - 7x^2 + 15x$

Second term of binomial times trinomial: $(3)(2x^2) + (3)(-7x) + (3)(15) = 6x^2 - 21x + 45$

Combine the results of both multiplications: $2x^3 - 7x^2 + 15x + 6x^2 - 21x + 45$

$-x^2$ $-6x$

Answer: $2x^3 - x^2 - 6x + 45$

MULTIPLYING TWO TRINOMIALS
We continue to multiply each term in the first polynomial by every other term in the second polynomial.

Example: Multiply $(2x^2 + 5x - 3)(3x^2 - 6x + 9)$

First term of first trinomial times second trinomial: $(2x^2)(3x^2) + (2x^2)(-6x) + (2x^2)(9) = 6x^4 - 12x^3 + 18x^2$

Second term of first trinomial times second trinomial: $(5x)(3x^2) + (5x)(-6x) + (5x)(9) = 15x^3 - 30x^2 + 45x$

Third term of first trinomial times second trinomial: $(-3)(3x^2) + (-3)(-6x) + (-3)(9) = -9x^2 + 18x - 27$

$3x^3$ $63x$

Combine all three products: $6x^4 - 12x^3 + 18x^2 + 15x^3 - 30x^2 + 45x - 9x^2 + 18x - 27$

$-21x^2$

Answer: $6x^4 + 3x^3 - 21x^2 + 63x - 27$

Practice:
Find the product.