Chapter 5

Systems of Equations

Section 5.1
Systems of Equations: Solve by Graphing

Systems of equations refer to the solution of two or more equations.

Because equations are represented by lines and these lines will eventually cross at a point (unless they are parallel), the solution to the system takes place when we find the point where the lines join.

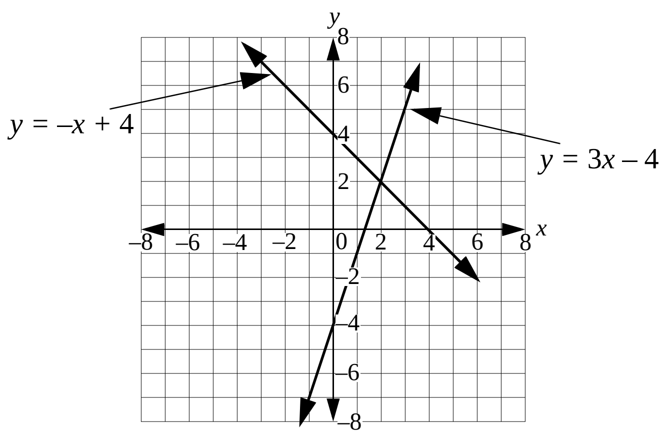
The most logical way to find the solution of a system is to plot the equations and read from the graph the coordinates (point) where the lines meet.

GRAPHICAL SOLUTION

Find the solution to the system of equations represented by $y = 3x - 4$ and $y = -x + 4$

First plot $y = 3x - 4$
where the y-intercept is (0,–4) and the slope (the coefficient of x) 3.

Second, plot $y = -x + 4$
where the y-intercept is (0,4) and the slope –1.



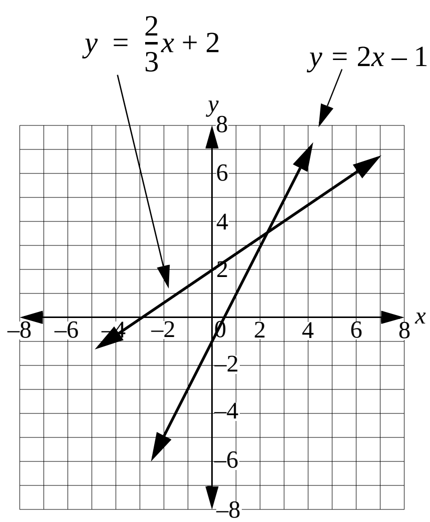
In the graph the lines cross at (2,2). This point is the solution to the system represented by the two equations given.

Graphical solutions have the drawback that the accuracy of the answer depends on how well you can plot the graph. Some systems may not be as simple as the one above.

Example: Find the solution to the system

$$y = 2x - 1$$
$$y = \frac{2}{3}x + 2$$

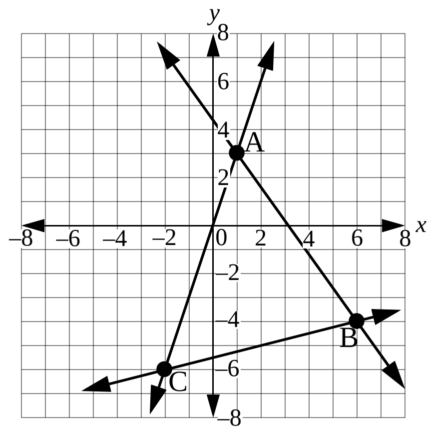
Plot the y-intercept and slope. The y-intercept for the first equation is at (0,–1) and the slope is 2. The y-intercept for the second equation is at (0,2) and the slope is $\frac{2}{3}$.



The graph shows the system plotted and the lines form a point at, approximately, (2.3, 3.6). Notice that it is difficult to define points that fall between integers, where exact answers are elusive.

To solve this dilemma, there are two other methods that may be used. One method is based on substituting one equation into the other, and the second method involves algebraic addition and substitution. Both are discussed in the next two sections, 5.2 and 5.3.

Example: Find the equations of the three lines that intersect at A(1,3), B(6,–4), C(–2,–6) to form triangle ABC.



Line AB

slope: $m = \dfrac{change\ \ of\ \ y_{\text{AB}}}{change\ \ of\ \ x_{\text{AB}}} = \dfrac{-7}{5}$ y-intercept: (0,4.3)

$y = mx + b$

$y = \dfrac{-7}{5}x + 4.3$

Line CB

slope: $m = \dfrac{change\ \ of\ \ y_{\text{CB}}}{change\ \ of\ \ x_{\text{CB}}} = \dfrac{2}{8} = \dfrac{1}{4}$ y-intercept: (0,–5.5)

$y = \dfrac{1}{4}x - 5.5$

Line CA

slope: $m = \dfrac{change\ \ of\ \ y_{\text{CA}}}{change\ \ of\ \ x_{\text{CA}}} = \dfrac{9}{3} = \dfrac{3}{1} = 3$ y-intercept: (0,0) is an empty value.

$y = 3x$ (empty)

Practice:

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Section 5.2
Systems of Equations: Solve by Substitution

THE SUBSTITUTION METHOD

Using the second example from section 5.1: $y = 2x - 1$ $\quad y = \frac{2}{3}x + 2$

We read these two equations and realize that the same variable, *y*, is equal to two distinct expressions.

because $y = y$

then $2x - 1 = \frac{2}{3}x + 2$

When we pair these two expressions, we get rid of *y* and are left with one *x* on each side. We then combine "like terms" and solve for *x*.

Solving a rational equation (an equation with fractions) is easier if we eliminate the fractions and turn them into integers. We do this by multiplying each term of the equation by 3 (because the 3 denominator is what makes it a fraction).

$$(3)2x - (3)1 = (3)\frac{2}{3}x + (3)2$$

The result is that we exchange cancelling the 3 denominator for a larger equation—of the same value—overall. The new equation is now:

$$6x - 3 = 2x + 6$$

$6x - 2x = 6 + 3$ Combine like terms

$4x = 9$ Divide both sides by 4

$$x = \frac{9}{4} = 2.25$$

The graphical solution in section 5.1 estimated $x = 2.3$, but $\mathbf{x = 2.25}$ is accurate and exact.

To find the *y* value of the equation, go back to either of the two original equations, substitute the value of *x*, and get the value for *y*.

$$y = 2(2.25) - 1$$
$$y = 4.5 - 1$$
$$y = 3.5$$

The point where the lines cross is (2.25, 3.5). Any system of equations can be solved by substitution.

Example: Solve the system by substitution $y = 2x + 7$

$y = 2x + 4$ Because $y = y$, substitute

$$2x + 7 = 2x + 4$$
$$2x - 2x = -7 + 4$$
$$0 \neq -3$$

Because 0 is not equal to –3, the lines will not meet and there is no solution to the system: The lines are parallel. If the solution had been a true statement, like –3 = –3, then there is only one solution (all the points are at the intersection) and both lines are the same (identity property).

Example: Solve the system by substitution

$$2x - 3y = 10$$
$$x + y = 2$$

To substitute, first solve one of the equations in terms of x or y. Solving for x, the second equation becomes:

$$x = -y + 2$$

Substituting $(-y + 2)$ for x into the first equation: $2(-y + 2) - 3y = 10$

Doing this gives us an equation without x.

Solving for y:

$2(-y + 2) - 3y = 10$	Multiply contents of parenthesis by 2
$-2y + 4 - 3y = 10$	Combine like terms
$-5y + 4 = 10$	Subtract 4 from both sides
$-5y + 4 - 4 = 10 - 4$	
$-5y = 6$	Divide both sides by -5
$y = \frac{6}{-5} = -1.2$	

Because $x = -y + 2$ and $y = -1.2$,

by substitution, then

$$x = -(-1.2) + 2$$
$$x = 1.2 + 2$$
$$x = 3.2$$

The solution to the system is point $(3.2, -1.2)$

Practice:

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Section 5.3
Systems of Equations: Solve by Addition

THE ADDITION METHOD

The "addition" method is actually the "elimination by algebraic addition" method. Where, instead of adding all the time, we also have to use the number-line approach of eliminating negatives with positives. The object of the approach is to either eliminate x to find y, or eliminate y to find x.

Example: Solve the following system by addition.

$$\begin{aligned} x - y &= 5 \\ x + y &= 8 \end{aligned}$$

First column, $x + x = 2x$

Second column $-y + y = 0$ (the ys are eliminated)

Third column $5 + 8 = 13$

$$\begin{aligned} x - y &= 5 \\ x + y &= 8 \\ \hline 2x \quad &= 13 \\ x &= \frac{13}{2} \\ x &= 6.5 \end{aligned}$$

Divide both sides by 2

As before, to find the value of y, substitute x into any of the two equations:

$$\begin{aligned} x + y &= 8 \\ 6.5 + y &= 8 \\ 6.5 - 6.5 + y &= 8 - 6.5 \\ y &= 8 - 6.5 \\ y &= 1.5 \end{aligned}$$

The solution to the system is point (6.5,1.5)

Example: Solve the following system by addition.

$$\begin{aligned} 2x + y &= 5 \\ x + y &= 3 \end{aligned}$$

If we add the columns the way they are, no elimination will take place; therefore, to eliminate the y variable, the second equation will change to $-x - y = -3$. We achieve this by multiplying both sides of the second equation by -1.

$$-1(x + y = 3) \quad \longrightarrow \quad -x - y = -3$$

Now add equations:

$$\begin{aligned} 2x + y &= 5 \\ -x - y &= -3 \\ \hline x \quad &= 2 \end{aligned}$$

If $x = 2$, then

$$\begin{aligned} 2 + y &= 3 \\ y &= 3 - 2 \\ y &= 1 \end{aligned}$$

The solution to the system is point (2,1).

Example: Solve the following system by addition.

$$5y = 12 - 2x$$
$$3x + 4y = 4$$

First, rearrange equation and place like terms in line (x above x and y above y)

$$2x + 5y = 12$$
$$3x + 4y = 4$$

This time the x will be eliminated by multiplying the first equation by 3 and the second equation by –2.

$$3(2x + 5y = 12) \rightarrow 6x + 15y = 36$$
$$-2(3x + 4y = 4) \rightarrow -6x - 8y = -8$$
$$7y = 28$$
$$y = \frac{28}{7}$$
$$y = 4$$

Substituting 4 for y in the second equation:

$$3x + 4(4) = 4$$
$$3x + 16 = 4$$
$$3x + 16 - 16 = 4 - 16$$
$$3x = -12$$
$$x = -\frac{12}{3}$$
$$x = -4$$

The solution to the system is point (–4,4)

Practice:

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Section 5.4
Systems of Equations and Problem Solving

Making quantitative decisions means that, after comparing numbers, we sometimes select the lowest price or perhaps the highest quantity, the fastest time, or maybe the shortest distance. But if we want the shortest time (for example, a trip) at the lowest price, could that be a problem? Sometimes selecting alternatives that are hard to match IS the problem. Setting up systems of equations that represent conflicting decision-making could go a long way in helping us solve quantitative problems.

Example:
Using the first example from section 3.6 and expanding it, we see that Pat needs to rent a truck for a few hours to move. However, this time she will have two competing alternatives, the original one where she would pay \$40 per day and \$0.50 per mile, and a new one where she only pays \$35 per day, but the cost per mile increases to \$0.54 per mile. Which one will it be, lower daily rate, or lower per mile rate?
Obviously the answer rests on the number of miles she plans to drive: Select lower "per day" alternative if the trip is short, or lower "per mile" alternative, if the trip is long. In any case, we must find that "miles" number where the total costs of both alternatives are the same.
We could solve the problem by trial and error, or we can write two linear equations that represent the conflicting alternatives and do it faster.

Alternative 1: \$0.50 per mile plus \$40 per day ➡ Cost = $0.50M + 40$ where M is miles
Alternative 2: \$0.54 per mile plus \$35 per day ➡ Cost = $0.54M + 35$

When costs are equal, what is the value of M?

$$0.50M + 40 = 0.54M + 35$$
$$0.50M - 0.54M = 35 - 40$$
$$-0.04M = -5$$
$$M = \frac{5}{0.04} = 125 \text{ miles}$$

Select alternative 1 if the trip is more than 125 miles and alternative 2 if the trip is less than 125 miles

Example:
Issac is 6 years older than Jose. Issac was three times as old as Jose two years ago. What are their ages now?

Two alternatives, two equations: One today, the other one two years ago.

Today: $I = J + 6$

Two years ago: $I - 2 = 3(J - 2)$
$$I = 3(J - 2) + 2$$
$$I = 3J - 6 + 2$$
$$I = 3J - 4$$

Solve equation for today and equation for two years ago as a system:
$$I = J + 6$$
$$I = 3J - 4$$
$$3J - 4 = J + 6$$
$$3J - J = 6 + 4$$
$$2J = 10$$
$$J = \frac{10}{2} = 5$$

If today Jose = 5, then Issac = 5 + 6 = 11

Example:
Two cars start a trip toward each other at the same time. One from Los Angeles, CA, to Miami, FL, another one from Miami to Los Angeles. The Miami car travels at 48 miles per hour, and the Los Angeles car at

52 miles per hour (average speed). If the distance between the two cities is 2,735 miles, how long will it take for the two cars to meet?

Because distance is a function of speed and time ($d = st$) and here time (t) is the same for both cars, solve for t.

$$t_{Miami} = \frac{d_M}{s_M}\ ,\quad t_{Los\ Angeles} = \frac{d_{LA}}{s_{LA}}\quad \text{and}\quad t_{Miami} = t_{LA}\quad \text{therefore} \rightarrow \frac{d_{LA}}{s_{LA}} = \frac{d_M}{s_M}\quad (1)$$

Because $d_{LA} + d_M = 2735$ then solve for d_M and substitute $d_M = 2735 - d_{LA}$ in equation (1)

Solving equations as a system $\frac{d_{LA}}{s_{LA}} = \frac{2735 - d_{LA}}{s_M}$ and substituting for s_{LA} and s_M $\frac{d_{LA}}{52} = \frac{2735 - d_{LA}}{48}$

Now cross-multiply and solve for d_{LA}:

$$48d_{LA} = 52(2735 - d_{LA})$$
$$48d_{LA} = 142220 - 52d_{LA}$$
$$52d_{LA} + 48d_{LA} = 142220$$
$$100d_{LA} = 142220$$
$$d_{LA} = \frac{142220}{100} = 1{,}422.2 \text{ miles}$$

If $d_{LA} = 1{,}422.2$ miles,

then $d_M = 2735 - 1422.2 = 1312.8$ miles

They would meet at 1,422.2 miles from Los Angeles, or 1,312.8 miles from Miami.

Time: $t_{LA} = \frac{1422.2}{52} = 27.35$ hours

Note on the use of “subscripts.”

The problem above uses “subscripts”, which are small letters placed slightly below a letter or number (d_{LA}). This letters have no numerical value and are there to clarify the name of a variable using the same letter, for example “time for Miami” is t_M and “time for Los Angeles” is t_{LA}. The same may be done for distance (d_M, d_{LA}) and speed (s_M, s_{LA}).

Example:

In a basketball game at school, 620 fans paid admission. If a student ticket was \$3 and general admission ticket \$8, how many students attended if \$2,405 were collected?

Two equations must be extracted from the information:

Students + General = 620 or S + G = 620 (1)

Students(3) + General(8) = 2405 or 3S + 8G = 2405 (2)

Solving system by algebraic addition:
Multiplying (1) by –3

$$\begin{aligned} 3S + 8G &= 2405 \\ -3S - 3G &= -1860 \\ \hline 5G &= 545 \end{aligned}$$

$$G = \frac{545}{5} = 109 \text{ fans, general admission}$$

620 – 109 = 511 students

Practice:

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Section 5.5
Graphing Systems of Inequalities

The solution to a system of equations is the point where the lines meet. The solution to a system of inequalities is not a point but a region surrounded by the lines the system of inequalities represents.

Example: Find the solution to the inequalities $x + y \geq 3$ and $y \leq x - 2$

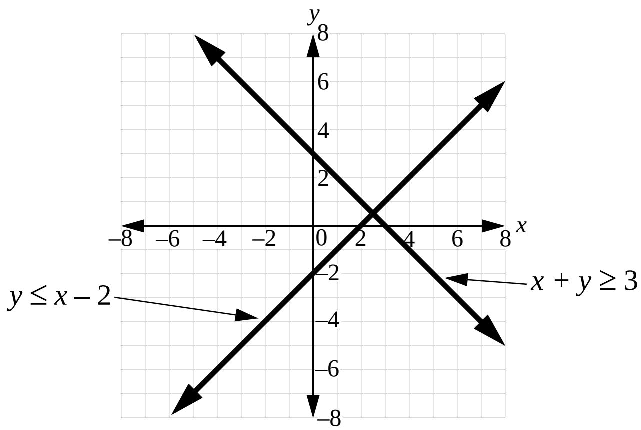
First, plot the inequalities using the same method used for equations:

The slope and *y*-intercept of the first inequality:

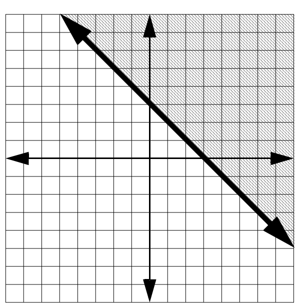
$$x + y \geq 3$$
$$x - x + y \geq -x + 3$$
$$y \geq -x + 3$$

The slope (coefficient of *x*) is –1 and the *y*-intercept is 3.

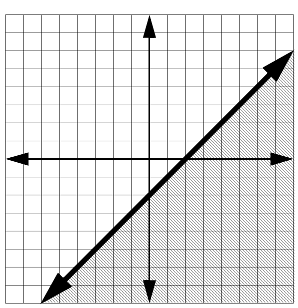
The slope of the second inequality is 1 (coefficient of *x*) and the *y*-intercept is –2. Both lines are plotted.



To find the region of the answer, look at the direction of the *y* inequalities only. In the first inequality, *y* is "greater than"; therefore, the solution is *above* (greater). The second inequality is "less than", thus the solution must be *below* the line. The graphs below show this for each line separately and then a third graph shows the solution (both combined).

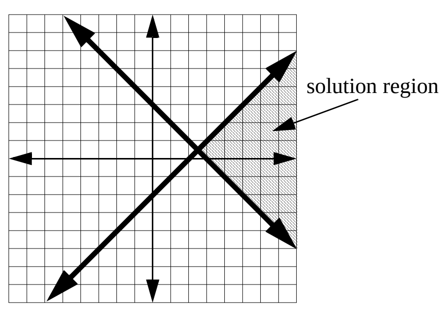


$y \geq -x + 3$



$y \leq x - 2$

Plotted separately

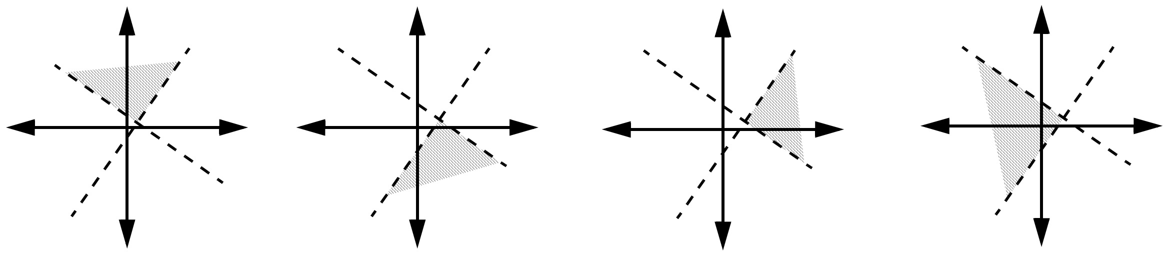


Solution region for both inequalities together. Every point in the solution region will work for both inequalities.

Solution regions for pairs of inequalities (positive slope and negative slope):

Examples:

y > y >	y < y <	y > y <	y < y >



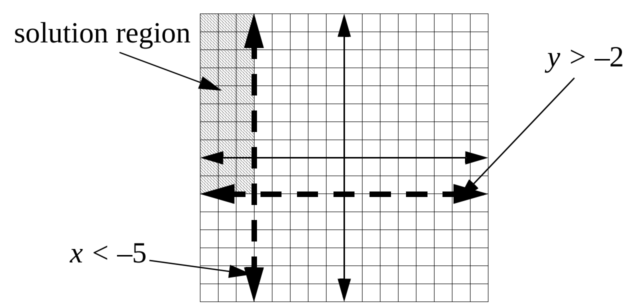
Both pos. and neg. slopes “greater than”	Both pos. and neg. slopes “less than”	Neg. slope “greater than” pos. slope “less than”	Pos. slope “greater than” neg. slope “less than”

Example: Find the solution region for $x < -5$ and $y > -2$

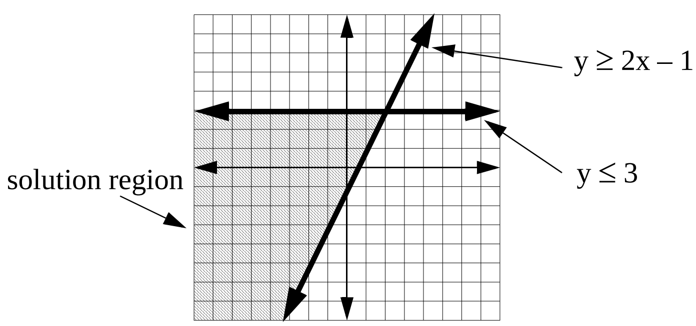
The first inequality is to the left of a vertical line
(*x* is “less” to the left.)

The second inequality is above a horizontal line
(*y* is “greater” over the horizontal.)

Because the inequalities are < and >, dashed lines apply.



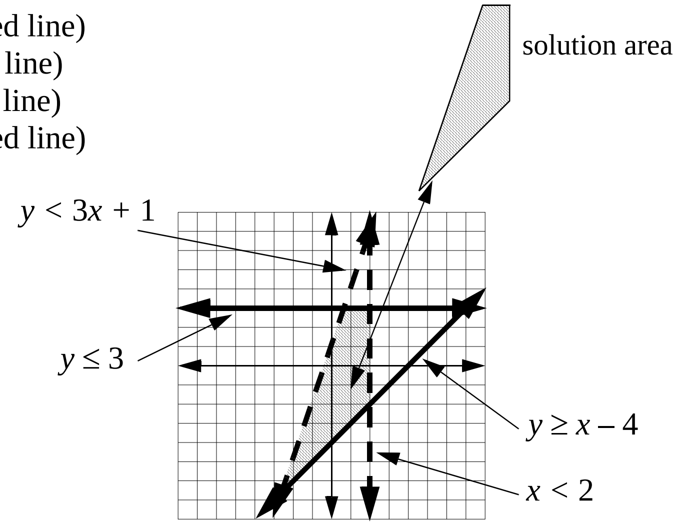
Example: Find the solution region for $y \le 3$ and $y \ge 2x - 1$



Because both lines are ≤ and ≥, solid lines apply.

Example: Find the solution area to the four inequalities:

$x < 2$	(dashed line)
$y \leq 3$	(solid line)
$y \geq x - 4$	(solid line)
$y < 3x + 1$	(dashed line)



Practice:

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